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Interior estimates for solutions of a fourth order nonlinear partial differential equation

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Abstract

Let $x:M\to A^{n+1}$ be a locally strongly convex hypersurface, given by the graph of a convex function $x_{n+1}=f(x_1,\dots,x_n)$ defined in a convex domain $\Omega\subset R^n$. M is called a α-extremal hypersurface, if f is a solution of

$$\sum U^{ij}\frac{\partial^2}{\partial x_i\partial x_j}\left(\left(\det\left(\frac{\partial^2 f}{\partial x_i\partial x_j}\right)\right)^{-\frac{n+1-\alpha}{n+2}}\right)=0,\quad \alpha\neq n+1.$$

The purpose of this paper is to establish an interior estimates for solutions of fourth order nonlinear PDE $\Delta\rho=\alpha\frac{\|\nabla\rho\|_G^2}{\rho}$ and prove that Euclidean complete α-extremal hypersurface must be an elliptic paraboloid for $|\alpha|\geqslant K(n)$, where $K(n)$ is a positive constant depending only on the dimension n.

MSC: 53A15

Keywords: Interior estimates; Uniqueness questions

Introduction

In this paper we study a nonlinear, fourth order, partial differential equation for a convex function f on a convex domain Ω in $\mathbb{R}^n$. The equation can be written as

$$\Delta\rho=\alpha\frac{\|\nabla\rho\|_G^2}{\rho}\tag{0.1}$$

where α is a constant, Δ denotes the Laplacian with respect to the Blaschke metric G of the graph of f in the $(n+1)$-dimensional affine space A^{n+1} with this graph considered as a differentiable hypersurface M immersed in

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A^{n+1} and

$$\rho=\left(\det\left(\frac{\partial f^2}{\partial x_i\partial x_j}\right)\right)^{-1/n+2}.$$

An equivalent form of the Eq. (0.1) is

$$\sum U^{ij}\frac{\partial^2}{\partial x_i\partial x_j}\left(\left(\det\left(\frac{\partial^2 f}{\partial x_i\partial x_j}\right)\right)^{-\frac{n+1-\alpha}{n+2}}\right)=0,\quad \alpha\neq n+1, \tag{0.2}$$

where (U^{ij}) is the matrix of cofactors of the matrix $(\frac{\partial^2 f}{\partial x_i\partial x_j})$.

For $\alpha=0$, the PDE (0.1) is the affine maximal hypersurface equation:

$$\sum U^{ij}\frac{\partial^2}{\partial x_i\partial x_j}\left(\left(\det\left(\frac{\partial^2 f}{\partial x_i\partial x_j}\right)\right)^{-\frac{n+1}{n+2}}\right)=0.$$

Let f be a strictly convex function defined for all $(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$. N. Trudinger and X.J. Wang have proved (see [1]) that if f satisfies the affine maximal hypersurface equation, then, for $n=2$, f must be a quadratic polynomial. For $n\geqslant 3$, whether f is a quadratic polynomial is still open.

For $n\geqslant 3$, Eq. (0.1) is the Euler–Lagrange equation of the functional

$$V(f,\Omega_0,\alpha)=\int\limits_{\Omega_0}\left(\det\left(\frac{\partial^2 f}{\partial x_i\partial x_j}(x)\right)\right)^{\frac{1+\alpha}{n+2}}dx_1\wedge\cdots\wedge dx_n,$$

where $\Omega_0\subset\Omega$ is a bounded domain.

For $\alpha=n+1$, Eq. (0.1) appears in the study of Einstein Kähler affine manifolds (see [3]).

Obviously, the convex function

$$f(x_1,x_2,\dots,x_n)=\frac{1}{2}\big((x_1)^2+(x_2)^2+\cdots+(x_n)^2\big),\quad (x_1,x_2,\dots,x_n)\in R^n$$

is a solution of the PDE (0.1) for every constant α.

In this paper we will be primarily interested in the case when $\alpha\neq 0$. Our purpose is to establish an interior estimates for solutions of fourth order nonlinear PDE (0.1), which can be applied to uniqueness questions for solutions of PDE (0.1). The results in this paper can be stated as follows:

Theorem 1. *Let $\Omega\subset\mathbb{R}^n$ be a bounded convex domain and $f\in C^4(\Omega)\cap C^0(\overline{\Omega})$ be an nonnegative strictly convex function, which satisfies*

$$\sum U^{ij}\frac{\partial^2}{\partial x_i\partial x_j}\left(\left(\det\left(\frac{\partial^2 f}{\partial x_i\partial x_j}\right)\right)^{-\frac{n+1-\alpha}{n+2}}\right)=0,\quad in\ \Omega,$$

with $f=T$ on $\partial\Omega$, where $\alpha\neq n+1$ and $T>0$ are constants. Then, there exists a positive constant $K=K(n)$ depending only on the dimension n such that, if $|\alpha|\geqslant K(n)$, then $f\in C^\infty(\Omega)$ and for any $0<T'<T$, we have the estimates

$$|D^k f|\leqslant C,\quad k=2,3,\dots,\ in\ \Omega_{T'}=\{x\in\Omega\mid f(x)<T'\},$$

where C is a constant depending only on T, T', k, $\operatorname{diam}(\overline{\Omega_{T'+\frac{T-T'}{2}}})$, $\operatorname{dist}(\overline{\Omega_{T'+\frac{T-T'}{2}}},\partial\Omega)$, α and the dimension n.

Theorem 2. *Let f be a strictly convex function defined for all $(x_1,\dots,x_n)\in\mathbb{R}^n$. Then, there exist a positive constant $K(n)$ depending only on the dimension n such that if $|\alpha|\geqslant K(n)$ and f satisfies*

$$\Delta\rho=\alpha\frac{\|\nabla\rho\|_G^2}{\rho},$$

then f must be a quadratic polynomial.

It is clear that Eq. (0.1) is invariant under unimodular affine transformations. The result in this paper will be proved in the language of equiaffine differential geometry. For the purpose of establishing the interior estimates for solutions of fourth order nonlinear PDE (0.1), we shall recall some fundamental formulas in equiaffine differential geometry. For the proofs of these formulas we refer to [4].

1. Fundamental formulas in equiaffine differential geometry

Let A^{n+1} be the real unimodular affine space of dimension $n+1$, M be a connected and oriented C^∞ manifold of dimension n, and $x : M \to A^{n+1}$ be a locally strongly convex hypersurface. Choose a local unimodular affine frame field $x; e_1, \dots, e_n, e_{n+1}$ on M such that

$$x \in M, e_1, \dots, e_n \in T_x M, \quad \det(e_1, \dots, e_n, e_{n+1}) = 1,$$
$$G_{ij} = \delta_{ij}, \quad e_{n+1} = Y,$$

where we denote by G_{ij} and Y the Blaschke metric and the equiaffine normal vector field, respectively. Denote by U, A_{ijk} and B_{ij} the affine equiconormal vector field, the Fubini–Pick tensor and the affine Weingarten tensor with respect to the frame field $e_1, \dots, e_n$, and by R_{ij} denotes the Ricci curvature. We have the following local formulas (see [4]):

$$x_{,ij} = \sum A_{ijk} e_k + G_{ij} Y, \tag{1.1}$$
$$U_{,ij} = -\sum A_{ijk} U_{,k} - B_{ij} U, \tag{1.2}$$
$$\Delta U = -n L_1 U, \tag{1.3}$$
$$\sum A_{iik} = 0 \quad (k = 1, 2, \dots, n), \tag{1.4}$$
$$R_{ij} = \sum A_{mli} A_{mlj} + \frac{n-2}{2} B_{ij} + \frac{n}{2} L_1 \delta_{ij}, \tag{1.5}$$

where L_1 denotes the equiaffine mean curvature, "Δ" and "," denote the Laplacian and the covariant differentiation with respect to the Blaschke metric G, respectively. Let $x : M \to A^{n+1}$ be given by a convex function

$$x_{n+1} = f(x_1, \dots, x_n)$$

defined in a convex domain $\Omega \subset \mathbb{R}^n$. We choose the following unimodular affine frame field:

$$e_1 = \left(1, 0, \dots, 0, \frac{\partial f}{\partial x_1}\right), \quad e_2 = \left(0, 1, \dots, 0, \frac{\partial f}{\partial x_2}\right), \quad \dots, \quad e_n = \left(0, 0, \dots, 1, \frac{\partial f}{\partial x_n}\right),$$
$$e_{n+1} = (0, 0, \dots, 0, 1).$$

Then the Blaschke metric is given by (see [4]):

$$G = \left(\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)\right)^{-1/(n+2)} \sum \frac{\partial^2 f}{\partial x_i \partial x_j} \, dx_i \, dx_j.$$

The equiaffine conormal vector field U can be identified with

$$U = \left(\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)\right)^{-1/(n+2)} \left(-\frac{\partial f}{\partial x_1}, \dots, -\frac{\partial f}{\partial x_n}, 1\right).$$

Denote

$$\rho := \left(\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)\right)^{-1/(n+2)}, \qquad \Phi := \frac{\|\nabla \rho\|_G^2}{\rho}.$$

By (1.3), we have

$$\Delta \rho = -n L_1 \rho. \tag{1.6}$$

The Laplacian with respect to the Blaschke metric is given by (see [2])

$$\Delta = \frac{1}{\rho}\sum f^{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \frac{n}{\rho^2}\sum f^{ij}\frac{\partial \rho}{\partial x_i}\frac{\partial}{\partial x_j}, \tag{1.7}$$

where (f^{ij}) denotes the inverse matrix of the matrix (f_{ij}) and $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$.

Consider the Legendre transformation relative to f:

$$\xi_i = \frac{\partial f}{\partial x_i}(x_1, \dots, x_n), \qquad u(\xi_1, \dots, \xi_n) = \sum x_i \frac{\partial f}{\partial x_i}(x_1, \dots, x_n) - f(x_1, \dots, x_n),$$

and denote by Ω^* the Legendre transformation domain of f, i.e. $u : \Omega^* \to \mathbb{R}$ and

$$\Omega^* = \bigl\{\bigl(\xi_1(x), \dots, \xi_n(x)\bigr) \mid x \in \Omega\bigr\}.$$

Note that, in this case, the Legendre transformation relative to f is a diffeomorphism. We have (see [2])

$$\Delta = \frac{1}{\rho}\sum u^{ij}\frac{\partial^2}{\partial \xi_i \partial \xi_j} - \frac{2}{\rho^2}\sum u^{ij}\frac{\partial \rho}{\partial \xi_i}\frac{\partial}{\partial \xi_j}, \tag{1.8}$$

where (u^{ij}) denotes the inverse matrix of the matrix (u_{ij}) and $u_{ij} = \frac{\partial^2 u}{\partial \xi_i \partial \xi_j}$.

2. Estimate for $\Delta\Phi$

In the following we assume that f is a solution of the forth order PDE:

$$\Delta\rho = \alpha\frac{\|\nabla\rho\|_G^2}{\rho}, \tag{2.1}$$

where α is a constant. Then we have

$$nL_1 = -\alpha\frac{\|\nabla\rho\|_G^2}{\rho^2}. \tag{2.2}$$

The estimate below follows analogously to the estimate in [8]. Let $p \in M$ be any fixed point. We choose a local orthonormal frame field of the Blaschke metric on M. Then

$$\Phi = \sum\frac{\rho_{,j}^2}{\rho}, \qquad \Phi_{,i} = 2\sum\frac{\rho_{,j}\rho_{,ji}}{\rho} - \rho_{,i}\sum\frac{\rho_{,j}^2}{\rho^2},$$

$$\Delta\Phi = 2\sum\frac{\rho_{,ji}^2}{\rho} + 2\sum\frac{\rho_{,j}\rho_{,jii}}{\rho} - 4\sum\frac{\rho_{,j}\rho_{,i}\rho_{,ji}}{\rho^2} + (2-\alpha)\frac{(\sum\rho_{,j}^2)^2}{\rho^3},$$

where we used (2.1). In the case $\Phi(p) = 0$, it is easy to get, at p,

$$\Delta\Phi \geqslant 2\sum\frac{\rho_{,ij}^2}{\rho}. \tag{2.3}$$

Now we assume that $\Phi(p) \neq 0$. Choose a local orthonormal frame field of the Blaschke metric on M such that $\rho_{,1}(p) = \|\nabla\rho\|_G(p) > 0$, $\rho_{,i}(p) = 0$ for all $i > 1$. Then

$$\Delta\Phi = 2\sum\frac{\rho_{,ij}^2}{\rho} + 2\sum\frac{\rho_{,j}\rho_{,jii}}{\rho} - 4\frac{\rho_{,1}^2\rho_{,11}}{\rho^2} + (2-\alpha)\frac{\rho_{,1}^4}{\rho^3}. \tag{2.4}$$

Applying the Schwarz's inequality we get

$$\begin{aligned}\sum\rho_{,ij}^2 &\geqslant \rho_{,11}^2 + \sum_{i>1}\rho_{,ii}^2 + 2\sum_{i>1}\rho_{,1i}^2 \geqslant \rho_{,11}^2 + \frac{1}{n-1}\Bigl(\sum_{i>1}\rho_{,ii}\Bigr)^2 + 2\sum_{i>1}\rho_{,1i}^2 \\ &\geqslant \frac{n}{n-1}\sum\rho_{,1i}^2 - 2\frac{\alpha}{n-1}\frac{\rho_{,1}^2\rho_{,11}}{\rho} + \frac{\alpha^2}{n-1}\frac{\rho_{,1}^4}{\rho^2}.\end{aligned} \tag{2.5}$$

An application of the Ricci identity shows that

$$\begin{aligned}2\sum\rho_{,j}\rho_{,jii}&=2\sum\rho_{,j}(\Delta\rho)_{,j}+2R_{11}\rho_{,1}^2\\&=2\sum\rho_{,j}\left(2\alpha\frac{\sum\rho_{,i}\rho_{,ij}}{\rho}-\alpha\frac{\rho_{,j}\sum\rho_{,i}^2}{\rho^2}\right)+2\sum A_{ml1}^2\rho_{,1}^2+(n-2)B_{11}\rho_{,1}^2+nL_1\rho_{,1}^2\\&\geqslant\big(4\alpha-(n-2)\big)\frac{\rho_{,1}^2\rho_{,11}}{\rho}+\left(-3\alpha-\frac{(n-2)^2(n-1)}{8n}\right)\frac{\rho_{,1}^4}{\rho^2}.\end{aligned}\tag{2.6}$$

Substituting (2.5) and (2.6) into (2.4) we obtain

$$\Delta\Phi\geqslant 2\delta\sum\frac{\rho_{,ij}^2}{\rho}+\frac{n(1-\delta)}{2(n-1)}\frac{\sum\Phi_{,i}^2}{\Phi}+a_0\sum\Phi_{,i}\frac{\rho_{,i}}{\rho}+b_0\frac{\Phi^2}{\rho},\tag{2.7}$$

where we denote

$$\begin{aligned}a_0&=2\frac{n-2+\delta}{n-1}\alpha-\frac{n^2+2n\delta-n-2}{2(n-1)},\\b_0&=\frac{2(1-\delta)}{n-1}\alpha^2-2\frac{n-\delta}{n-1}\alpha+2-\frac{n^2+n\delta-2}{2(n-1)}-\frac{(n-2)^2(n-1)}{8n}.\end{aligned}$$

3. Estimate for $g^2\rho\frac{\|\nabla f\|_G^2}{(1+f)^2}$

Suppose that $x:M\to A^{n+1}$ is a locally strongly convex hypersurface, given as the graph of a solution f of PDE (2.1) defined for all $(x_1,\dots,x_n)\in\mathbb{R}^n$. Let $p\in M$ be any fixed point. By an unimodular affine transformation, we may assume that p has coordinates $(0,\dots,0)$ and

$$f(0)=0,\qquad\frac{\partial f}{\partial x_i}(0)=0,\qquad\text{for all } i\geqslant 1.$$

With respect to this coordinate system, we have $f\geqslant 0$. Since M is Euclidean complete, for any number $C>0$, the set

$$M_C=\big\{\big(x_1,\dots,x_n,f(x_1,\dots,x_n)\big)\in M\mid f(x_1,\dots,x_n)\leqslant C\big\}$$

is compact.

Proposition 3.1. *We have the following estimate for* $g^2\rho\frac{\|\nabla f\|_G^2}{(1+f)^2}$

$$\begin{aligned}&\exp\left\{\frac{-m}{\ln(1+C)-\ln(1+f)}\right\}g^2\rho\frac{\|\nabla f\|_G^2}{(1+f)^2}\\&\quad\leqslant\exp\{-\ln\cos(2\epsilon)\}\frac{1}{a}\left(\frac{(n+2)^2}{8\delta\epsilon\tan(\epsilon)}(\widetilde{A}+1)+2n+b\widetilde{A}-\tan(2\epsilon)b_0'(\alpha)\widetilde{A}\right)\\&\qquad+\exp\{-\ln\cos(2\epsilon)\}\left(\frac{1}{a}\left(16ce^{-4}\frac{1}{m^2}+2916n^2e^{-6}\frac{1}{m^4}\right)\right)^{\frac12},\end{aligned}$$

where ϵ, a, b, c and $b_0'(\alpha)$ are constants. ϵ, a, b and c depend only on the dimension n and $b_0'(\alpha)$ depends additionally α,

$$0<\delta<\frac{1}{3n-2},\quad m=\frac{80(1+\frac{1}{n-1}-\delta\frac{3n-2}{2(n-1)})\ln(1+C)}{\frac{1}{n-1}-\delta\frac{3n-2}{2(n-1)}},\quad\widetilde{F}=\exp\left\{\frac{-m}{\ln(1+C)-\ln(1+f)}\right\}\Phi,\quad\widetilde{A}=\max_{M_C}\widetilde{F}.$$

Proof. We consider the function:

$$F=\exp\left\{\frac{-m}{\ln(1+C)-\ln(1+f)}+\Psi\right\}\frac{m^2}{(\ln(1+C)-\ln(1+f))^4}\rho\frac{\|\nabla f\|_G^2}{(1+f)^2}\tag{3.1}$$

defined on M_C, where

$$\Psi = -\ln\cos\left(\epsilon + \frac{1}{A}\exp\left\{\frac{-m}{\ln(1+C)-\ln(1+f)}\right\}\Phi\right),$$
$$A = \eta\left(\max_{M_C}\exp\left\{\frac{-m}{\ln(1+C)-\ln(1+f)}\right\}\Phi + 1\right),$$

ϵ, η and m are positive constants to be determined later. Clearly, F attains its supremum at some interior point p^* of M_C. We can assume that $\|\nabla f\|_G(p^*) > 0$. Choose a local orthonormal frame field of the Blaschke metric on M such that $f_{,1}(p^*) = \|\nabla f\|_G(p^*) > 0$, $f_{,i}(p^*) = 0$ for all $i > 1$. We have, at p^*,

$$F_{,i} = 0, \tag{3.2}$$
$$\sum F_{,ii} \leqslant 0. \tag{3.3}$$

We now calculate both expressions (3.2) and (3.3) explicitly. To simplify the expressions we denote

$$g = \frac{m}{(\ln(1+C)-\ln(1+f))^2}, \qquad g' = \frac{m}{(\ln(1+C)-\ln(1+f))^3}.$$

By (3.2) and (3.3), we have

$$2\sum f_{,j}f_{,ji} + \left(-g\frac{f_{,i}}{1+f} - 2\frac{f_{,i}}{1+f} + 4\frac{g'}{g}\frac{f_{,i}}{1+f} + \frac{\rho_{,i}}{\rho} + \Psi_{,i}\right)\sum f_{,j}^2 = 0, \tag{3.4}$$

$$\begin{aligned}
&2\sum f_{,ij}^2 + 2\sum f_{,j}f_{,jii} + 2\sum\left(-g\frac{f_{,i}}{1+f} - 2\frac{f_{,i}}{1+f} + 4\frac{g'}{g}\frac{f_{,i}}{1+f} + \frac{\rho_{,i}}{\rho} + \Psi_{,i}\right)f_{,j}f_{,ji}\\
&\quad + \left(-2g'\sum\frac{f_{,i}^2}{(1+f)^2} - g\frac{\Delta f}{1+f} + g\sum\frac{f_{,i}^2}{(1+f)^2} + 4\left(\frac{g'}{g}\right)^2\sum\frac{f_{,i}^2}{(1+f)^2}\right)\sum f_{,j}^2\\
&\quad + \left(-4\frac{g'}{g}\sum\frac{f_{,i}^2}{(1+f)^2} + 4\frac{g'}{g}\frac{\Delta f}{1+f} + 2\sum\frac{f_{,i}^2}{(1+f)^2} - 2\frac{\Delta f}{1+f} - \sum\frac{\rho_{,i}^2}{\rho^2} + \frac{\Delta\rho}{\rho} + \Delta\Psi\right)\sum f_{,j}^2 \leqslant 0.
\end{aligned} \tag{3.5}$$

Let us simplify (3.5). Inserting (3.4) into (3.5) and noting

$$2\sum f_{,ij}^2 \geqslant 2f_{,11}^2 + 2\sum_{i>1} f_{,ii}^2 + 4\sum_{i>1} f_{,1i}^2 \geqslant 2\left(\frac{n}{n-1} - \delta\right)f_{,11}^2 + 4\sum_{i>1} f_{,1i}^2 - \frac{2-2\delta(n-1)}{\delta(n-1)^2}(\Delta f)^2,$$

for any $0 < \delta < 1$, we get

$$\begin{aligned}
&\left(2\left(\frac{n}{n-1} - \delta\right) - 4\right)f_{,11}^2 + 2\sum f_{,j}f_{,jii} - \frac{2-2\delta(n-1)}{\delta(n-1)^2}(\Delta f)^2\\
&\quad + \left(-2g'\frac{f_{,1}^2}{(1+f)^2} - g\frac{\Delta f}{1+f} + g\frac{f_{,1}^2}{(1+f)^2} + 4\left(\frac{g'}{g}\right)^2\frac{f_{,1}^2}{(1+f)^2}\right)f_{,1}^2\\
&\quad + \left(-4\frac{g'}{g}\frac{f_{,1}^2}{(1+f)^2} + 4\frac{g'}{g}\frac{\Delta f}{1+f} + 2\frac{f_{,1}^2}{(1+f)^2} - 2\frac{\Delta f}{1+f} - \frac{\Phi}{\rho} + \frac{\Delta\rho}{\rho} + \Delta\Psi\right)f_{,1}^2 \leqslant 0.
\end{aligned} \tag{3.6}$$

We now compute the term $\sum f_{,j}f_{,jii}$. An application of the Ricci identity shows that

$$\begin{aligned}
2\sum f_{,j}f_{,jii} &= 2\sum f_{,j}(\Delta f)_{,j} + 2\sum R_{ij}f_{,i}f_{,j} = 2\sum A_{ml1}^2 f_{,1}^2 - (n+2)B_{11}f_{,1}^2 + nL_1 f_{,1}^2\\
&\geqslant (2-2\delta)\sum A_{ml1}^2 f_{,1}^2 + (n+2)\frac{\rho_{,11}}{\rho}f_{,1}^2 - \frac{(n+2)^2}{8\delta}\frac{\Phi}{\rho}f_{,1}^2 - \frac{\Delta\rho}{\rho}f_{,1}^2\\
&\geqslant (2-2\delta)\left(\frac{n}{n-1} - \delta\right)f_{,11}^2 - (1-\delta)\left(\frac{2-2\delta(n-1)}{\delta(n-1)^2} + \frac{2}{n}\right)(\Delta f)^2\\
&\quad + (n+2)\frac{\rho_{,11}}{\rho}f_{,1}^2 - \frac{(n+2)^2}{8\delta}\frac{\Phi}{\rho}f_{,1}^2 - \frac{\Delta\rho}{\rho}f_{,1}^2.
\end{aligned} \tag{3.7}$$

Substituting (3.7) into (3.6) we obtain

$$\begin{aligned}
&\left(\frac{1}{n-1}-\delta\frac{3n-2}{2(n-1)}\right)\left(g\frac{f_{,1}}{1+f}+2\frac{f_{,1}}{1+f}-4\frac{g'}{g}\frac{f_{,1}}{1+f}-\frac{\rho_{,1}}{\rho}-\Psi_{,1}\right)^2 f_{,1}^2\\
&\quad-\left(1+\frac{(n+2)^2}{8\delta}\right)\frac{\Phi}{\rho}f_{,1}^2-\left((2-\delta)\frac{2-2\delta(n-1)}{\delta(n-1)^2}+\frac{2-2\delta}{n}\right)(\Delta f)^2+(n+2)\frac{\rho_{,11}}{\rho}f_{,1}^2\\
&\quad+\left(-2g'\frac{f_{,1}^2}{(1+f)^2}-g\frac{\Delta f}{1+f}+g\frac{f_{,1}^2}{(1+f)^2}+4\left(\frac{g'}{g}\right)^2\frac{f_{,1}^2}{(1+f)^2}\right)f_{,1}^2\\
&\quad+\left(-4\frac{g'}{g}\frac{f_{,1}^2}{(1+f)^2}+4\frac{g'}{g}\frac{\Delta f}{1+f}+2\frac{f_{,1}^2}{(1+f)^2}-2\frac{\Delta f}{1+f}+\Delta\Psi\right)f_{,1}^2\leqslant 0.
\end{aligned}\tag{3.8}$$

Now we calculate $\Delta\Psi$. We have

$$\Psi_{,i}=\frac{1}{A}\tan\left(\epsilon+\frac{\widetilde{F}}{A}\right)\widetilde{F}_{,i},\tag{3.9}$$

where $\widetilde{F}=\exp\{\frac{-m}{\ln(1+C)-\ln(1+f)}\}\Phi$. Note that

$$\begin{aligned}
&\frac{1}{A}\tan\left(\epsilon+\frac{\widetilde{F}}{A}\right)\exp\left\{\frac{-m}{\ln(1+C)-\ln(1+f)}\right\}\left(2g\sum\frac{f_{,i}}{1+f}\left(\Phi_{,i}-g\frac{f_{,i}}{1+f}\Phi\right)\right)\\
&\quad\leqslant\frac{1}{A^2}\sum\left(\exp\left\{\frac{-m}{\ln(1+C)-\ln(1+f)}\right\}\left(\Phi_{,i}-g\frac{f_{,i}}{1+f}\Phi\right)\right)^2+\tan^2\left(\epsilon+\frac{\widetilde{F}}{A}\right)g^2\sum\frac{f_{,i}^2}{(1+f)^2}.
\end{aligned}$$

Hence

$$\begin{aligned}
\Delta\Psi&=\frac{1}{A^2}\left(1+\tan^2\left(\epsilon+\frac{\widetilde{F}}{A}\right)\right)\sum\widetilde{F}_{,i}^2+\frac{1}{A}\tan\left(\epsilon+\frac{\widetilde{F}}{A}\right)\Delta\widetilde{F}\\
&=\frac{1}{A^2}\left(1+\tan^2\left(\epsilon+\frac{\widetilde{F}}{A}\right)\right)\sum\left(\exp\left\{\frac{-m}{\ln(1+C)-\ln(1+f)}\right\}\left(\Phi_{,i}-g\frac{f_{,i}}{1+f}\Phi\right)\right)^2\\
&\quad+\frac{1}{A}\tan\left(\epsilon+\frac{\widetilde{F}}{A}\right)\exp\left\{\frac{-m}{\ln(1+C)-\ln(1+f)}\right\}\left(\Delta\Phi+g^2\sum\frac{f_{,i}^2}{(1+f)^2}\Phi\right.\\
&\quad\left.-2g'\sum\frac{f_{,i}^2}{(1+f)^2}\Phi-g\frac{\Delta f}{1+f}\Phi+g\sum\frac{f_{,i}^2}{(1+f)^2}\Phi-2g\sum\frac{f_{,i}\Phi_{,i}}{1+f}\right)\\
&\geqslant\frac{1}{A^2}\tan^2\left(\epsilon+\frac{\widetilde{F}}{A}\right)\sum\widetilde{F}_{,i}^2-\tan^2\left(\epsilon+\frac{\widetilde{F}}{A}\right)g^2\frac{f_{,1}^4}{(1+f)^2}\\
&\quad+\frac{1}{A}\tan\left(\epsilon+\frac{\widetilde{F}}{A}\right)\exp\left\{\frac{-m}{\ln(1+C)-\ln(1+f)}\right\}\left(\Delta\Phi-g^2\frac{f_{,1}^2}{(1+f)^2}\Phi\right.\\
&\quad\left.+g\frac{f_{,1}^2}{(1+f)^2}\Phi-2g'\frac{f_{,1}^2}{(1+f)^2}\Phi-g\frac{\Delta f}{1+f}\Phi\right)f_{,1}^2.
\end{aligned}\tag{3.10}$$

Applying the Schwarz's inequality, we get

$$\begin{aligned}
(n+2)\frac{\rho_{,11}}{\rho}f_{,1}^2&\leqslant\frac{1}{A}\tan\left(\epsilon+\frac{\widetilde{F}}{A}\right)\exp\left\{\frac{-m}{\ln(1+C)-\ln(1+f)}\right\}2\delta\sum\frac{\rho_{,ij}^2}{\rho}f_{,1}^2\\
&\quad+\frac{(n+2)^2}{8\delta}\frac{A}{\tan(\epsilon+\frac{\widetilde{F}}{A})}\exp\left\{\frac{m}{\ln(1+C)-\ln(1+f)}\right\}\frac{1}{\rho}f_{,1}^2.
\end{aligned}\tag{3.11}$$

To simplify expressions we denote

$$a_1=\frac{1}{n-1}-\delta\frac{3n-2}{2(n-1)},\qquad a_2=1+\frac{(n+2)^2}{8\delta},\qquad a_3=(2-\delta)\frac{2-2\delta(n-1)}{\delta(n-1)^2}+\frac{2-2\delta}{n}.$$

Recall that $\Psi_{,i}=\frac{1}{A}\tan(\epsilon+\frac{\widetilde{F}}{A})\widetilde{F}_{,i}$. Then we have

$$\begin{aligned}&a_1\left(g\frac{f_{,1}}{1+f}+2\frac{f_{,1}}{1+f}-4\frac{g'}{g}\frac{f_{,1}}{1+f}-\frac{\rho_{,1}}{\rho}-\Psi_{,1}\right)^2 f_{,1}^2+\frac{1}{A^2}\tan^2\left(\epsilon+\frac{\widetilde{F}}{A}\right)\sum\widetilde{F}_{,i}^2\\&\quad\geqslant\frac{9}{10}\frac{a_1}{1+a_1}\left(g+2-4\frac{g'}{g}\right)^2\frac{f_{,1}^4}{(1+f)^2}-9\frac{a_1}{1+a_1}\frac{\Phi}{\rho}f_{,1}^2.\end{aligned}\tag{3.12}$$

Inserting (3.10), (3.11) and (3.12) into (3.8) we get

$$\begin{aligned}&\frac{9}{10}\frac{a_1}{1+a_1}\left(g+2-4\frac{g'}{g}\right)^2\frac{f_{,1}^4}{(1+f)^2}-\left(a_2+9\frac{a_1}{1+a_1}\right)\frac{\Phi}{\rho}f_{,1}^2-a_3(\Delta f)^2\\&\quad-\frac{(n+2)^2}{8\delta}\frac{A}{\tan(\epsilon+\frac{\widetilde{F}}{A})}\exp\left\{\frac{m}{\ln(1+C)-\ln(1+f)}\right\}\frac{1}{\rho}f_{,1}^2-\tan^2\left(\epsilon+\frac{\widetilde{F}}{A}\right)g^2\frac{f_{,1}^4}{(1+f)^2}\\&\quad+\left(-2g'\frac{f_{,1}^2}{(1+f)^2}-g\frac{\Delta f}{1+f}+g\frac{f_{,1}^2}{(1+f)^2}+4\left(\frac{g'}{g}\right)^2\frac{f_{,1}^2}{(1+f)^2}\right)f_{,1}^2\\&\quad+\left(-4\frac{g'}{g}\frac{f_{,1}^2}{(1+f)^2}+4\frac{g'}{g}\frac{\Delta f}{1+f}+2\frac{f_{,1}^2}{(1+f)^2}-2\frac{\Delta f}{1+f}\right)f_{,1}^2\\&\quad+\frac{\widetilde{F}}{A}\tan\left(\epsilon+\frac{\widetilde{F}}{A}\right)\left(-g^2\frac{f_{,1}^2}{(1+f)^2}+g\frac{f_{,1}^2}{(1+f)^2}-2g'\frac{f_{,1}^2}{(1+f)^2}-g\frac{\Delta f}{1+f}\right)f_{,1}^2\\&\quad+\frac{1}{A}\tan\left(\epsilon+\frac{\widetilde{F}}{A}\right)\exp\left\{\frac{-m}{\ln(1+C)-\ln(1+f)}\right\}\left(\Delta\Phi-2\delta\frac{\sum\rho_{,ij}^2}{\rho}\right)f_{,1}^2\leqslant 0.\end{aligned}\tag{3.13}$$

Multiply to both sides of (3.13) by $\frac{\rho^2}{(1+f)^2}$. Then we obtain

$$\begin{aligned}&\frac{9}{10}\frac{a_1}{1+a_1}\left(g+2-4\frac{g'}{g}\right)^2\rho^2\frac{f_{,1}^4}{(1+f)^4}-\left(a_2+9\frac{a_1}{1+a_1}\right)\Phi\rho\frac{f_{,1}^2}{(1+f)^2}\\&\quad-a_3(\Delta f)^2\frac{\rho^2}{(1+f)^2}-\tan^2\left(\epsilon+\frac{\widetilde{F}}{A}\right)g^2\rho^2\frac{f_{,1}^4}{(1+f)^4}\\&\quad-\frac{(n+2)^2}{8\delta}\frac{A}{\tan(\epsilon+\frac{\widetilde{F}}{A})}\exp\left\{\frac{m}{\ln(1+C)-\ln(1+f)}\right\}\rho\frac{f_{,1}^2}{(1+f)^2}\\&\quad+\left(-2g'\frac{f_{,1}^2}{(1+f)^2}-g\frac{\Delta f}{1+f}+g\frac{f_{,1}^2}{(1+f)^2}+4\left(\frac{g'}{g}\right)^2\frac{f_{,1}^2}{(1+f)^2}\right)\rho^2\frac{f_{,1}^2}{(1+f)^2}\\&\quad+\left(-4\frac{g'}{g}\frac{f_{,1}^2}{(1+f)^2}+4\frac{g'}{g}\frac{\Delta f}{1+f}+2\frac{f_{,1}^2}{(1+f)^2}-2\frac{\Delta f}{1+f}\right)\rho^2\frac{f_{,1}^2}{(1+f)^2}\\&\quad+\frac{\widetilde{F}}{A}\tan\left(\epsilon+\frac{\widetilde{F}}{A}\right)\left(-g^2\frac{f_{,1}^2}{(1+f)^2}+g\frac{f_{,1}^2}{(1+f)^2}-2g'\frac{f_{,1}^2}{(1+f)^2}-g\frac{\Delta f}{1+f}\right)\rho^2\frac{f_{,1}^2}{(1+f)^2}\\&\quad+\frac{1}{A}\tan\left(\epsilon+\frac{\widetilde{F}}{A}\right)\exp\left\{\frac{-m}{\ln(1+C)-\ln(1+f)}\right\}\left(\Delta\Phi-2\delta\frac{\sum\rho_{,ij}^2}{\rho}\right)\rho^2\frac{f_{,1}^2}{(1+f)^2}\leqslant 0.\end{aligned}\tag{3.14}$$

We choose the following values for δ and m:

$$0<\delta<\frac{1}{3n-2},\qquad m=\frac{80(1+\frac{1}{n-1}-\delta\frac{3n-2}{2(n-1)})\ln(1+C)}{\frac{1}{n-1}-\delta\frac{3n-2}{2(n-1)}}.$$

To further estimate expression (3.14), we use the formula $\Delta f = \frac{n}{\rho} + \frac{n}{\rho}\sum \rho_{,i} f_{,i}$ and the definition of g' and g; we have the inequalities:

$$-2g'\rho^2\frac{f_{,1}^4}{(1+f)^4} \geqslant -\frac{a_1}{10(1+a_1)}\left(g+2-4\frac{g'}{g}\right)^2\rho^2\frac{f_{,1}^4}{(1+f)^4},$$

$$-g\frac{\Delta f}{1+f}\rho^2\frac{f_{,1}^2}{(1+f)^2} \geqslant -\frac{a_1}{5(1+a_1)}\left(g+2-4\frac{g'}{g}\right)^2\rho^2\frac{f_{,1}^4}{(1+f)^4} - n^2\frac{10(1+a_1)}{a_1} - n^2\frac{10(1+a_1)}{a_1}\Phi\rho\frac{f_{,1}^2}{(1+f)^2},$$

$$-2\frac{\Delta f}{1+f}\rho^2\frac{f_{,1}^2}{(1+f)^2} \geqslant -2n\rho\frac{f_{,1}^2}{(1+f)^2} - \rho^2\frac{f_{,1}^4}{(1+f)^4} - n^2\Phi\rho\frac{f_{,1}^2}{(1+f)^2},$$

$$(\Delta f)^2\frac{\rho^2}{(1+f)^2} \leqslant 2n^2 + 2n^2\Phi\rho\frac{f_{,1}^2}{(1+f)^2},$$

$$4\frac{g'}{g}\frac{\Delta f}{1+f}\rho^2\frac{f_{,1}^2}{(1+f)^2} \geqslant -\rho^2\frac{f_{,1}^4}{(1+f)^4} - \frac{4n^2}{(\ln(1+C)-\ln(1+f))^2} - 4\left(\frac{g'}{g}\right)^2\rho^2\frac{f_{,1}^4}{(1+f)^4} - n^2\Phi\rho\frac{f_{,1}^2}{(1+f)^2},$$

$$g - 4\frac{g'}{g} \geqslant 0.$$

Substituting these inequalities into (3.14) we get

$$\begin{aligned}
&\frac{3a_1}{5(1+a_1)}\left(g+2-4\frac{g'}{g}\right)^2\rho^2\frac{f_{,1}^4}{(1+f)^4} \\
&\quad - \left(a_2 + 9\frac{a_1}{1+a_1} + n^2\frac{10(1+a_1)}{a_1} + 2n^2 + 2n^2a_3\right)\Phi\rho\frac{f_{,1}^2}{(1+f)^2} \\
&\quad - 2n\rho\frac{f_{,1}^2}{(1+f)^2} - \left(n^2\frac{10(1+a_1)}{a_1} + 2n^2a_3 + \frac{4n^2}{(\ln(1+C)-\ln(1+f))^2}\right) \\
&\quad - \tan^2\left(\epsilon + \frac{\widetilde{F}}{A}\right)g^2\rho^2\frac{f_{,1}^4}{(1+f)^4} - \frac{(n+2)^2}{8\delta}\frac{A}{\tan(\epsilon+\frac{\widetilde{F}}{A})}\exp\left\{\frac{m}{\ln(1+C)-\ln(1+f)}\right\}\rho\frac{f_{,1}^2}{(1+f)^2} \\
&\quad + \frac{\widetilde{F}}{A}\tan\left(\epsilon+\frac{\widetilde{F}}{A}\right)\left(-g^2\frac{f_{,1}^2}{(1+f)^2} + g\frac{f_{,1}^2}{(1+f)^2} - 2g'\frac{f_{,1}^2}{(1+f)^2} - g\frac{\Delta f}{1+f}\right)\rho^2\frac{f_{,1}^2}{(1+f)^2} \\
&\quad + \frac{1}{A}\tan\left(\epsilon+\frac{\widetilde{F}}{A}\right)\exp\left\{\frac{-m}{\ln(1+C)-\ln(1+f)}\right\}\left(\Delta\Phi - 2\delta\frac{\sum\rho_{,ij}^2}{\rho}\right)\rho^2\frac{f_{,1}^2}{(1+f)^2} \leqslant 0.
\end{aligned} \tag{3.15}$$

We shall consider different cases according to the values of $\Phi(p^*)$.

Case 1: $\Phi(p^*) = 0$.

Case 2: $\Phi(p^*) \neq 0$.

Case 1: By (2.3) we have

$$\Delta\Phi \geqslant 2\frac{\sum\rho_{,ij}^2}{\rho}.$$

Then by (3.15)

$$\frac{3a_1}{5(1+a_1)}\left(g+2-4\frac{g'}{g}\right)^2\rho^2\frac{f_{,1}^4}{(1+f)^4}$$

$$
\begin{aligned}
&-\left(a_2+9\frac{a_1}{1+a_1}+n^2\frac{10(1+a_1)}{a_1}+2n^2+2n^2a_3\right)\Phi\rho\frac{f_{,1}^2}{(1+f)^2}\\
&-2n\rho\frac{f_{,1}^2}{(1+f)^2}-\left(n^2\frac{10(1+a_1)}{a_1}+2n^2a_3+\frac{4n^2}{(\ln(1+C)-\ln(1+f))^2}\right)\\
&-\tan^2\left(\epsilon+\frac{\widetilde{F}}{A}\right)g^2\rho^2\frac{f_{,1}^4}{(1+f)^4}-\frac{(n+2)^2}{8\delta}\frac{A}{\tan(\epsilon+\frac{\widetilde{F}}{A})}\exp\left\{\frac{m}{\ln(1+C)-\ln(1+f)}\right\}\rho\frac{f_{,1}^2}{(1+f)^2}\leqslant 0. \quad (3.16)
\end{aligned}
$$

Case 2: We have, by (2.7),

$$
\Delta\Phi\geqslant 2\delta\frac{\sum\rho_{,ij}^2}{\rho}+b_0'(\alpha)\frac{\Phi^2}{\rho},
$$

where we denote

$$
\begin{aligned}
b_0'(\alpha)=&\left(\frac{2(1-\delta)}{n-1}-3\frac{(n-2+\delta)^2}{n(n-1)(1-\delta)}\right)\alpha^2-2\frac{n-\delta}{n-1}\alpha+2-\frac{n^2+n\delta-2}{2(n-1)}\\
&-\frac{(n-2)^2(n-1)}{8n}-\frac{3}{8}\frac{(n^2-n(1-2\delta)-2)^2}{n(n-1)(1-\delta)}.
\end{aligned}
$$

Then by (3.15) we have

$$
\begin{aligned}
&\frac{3a_1}{5(1+a_1)}\left(g+2-4\frac{g'}{g}\right)^2\rho^2\frac{f_{,1}^4}{(1+f)^4}\\
&-\left(a_2+9\frac{a_1}{1+a_1}+n^2\frac{10(1+a_1)}{a_1}+2n^2+2n^2a_3\right)\Phi\rho\frac{f_{,1}^2}{(1+f)^2}\\
&-2n\rho\frac{f_{,1}^2}{(1+f)^2}-\left(n^2\frac{10(1+a_1)}{a_1}+2n^2a_3+\frac{4n^2}{(\ln(1+C)-\ln(1+f))^2}\right)\\
&-\tan^2\left(\epsilon+\frac{\widetilde{F}}{A}\right)g^2\rho^2\frac{f_{,1}^4}{(1+f)^4}-\frac{(n+2)^2}{8\delta}\frac{A}{\tan(\epsilon+\frac{\widetilde{F}}{A})}\exp\left\{\frac{m}{\ln(1+C)-\ln(1+f)}\right\}\rho\frac{f_{,1}^2}{(1+f)^2}\\
&+\frac{\widetilde{F}}{A}\tan\left(\epsilon+\frac{\widetilde{F}}{A}\right)\left(-2g^2\rho^2\frac{f_{,1}^4}{(1+f)^4}+g\rho^2\frac{f_{,1}^4}{(1+f)^4}-\frac{a_1}{20(1+a_1)}g^2\rho^2\frac{f_{,1}^4}{(1+f)^4}\right.\\
&\left.-n^2\frac{10(1+a_1)}{a_1}-n^2\frac{10(1+a_1)}{a_1}\Phi\rho\frac{f_{,1}^2}{(1+f)^2}\right)\\
&+\frac{\widetilde{F}}{A}\tan\left(\epsilon+\frac{\widetilde{F}}{A}\right)b_0'(\alpha)\Phi\rho\frac{f_{,1}^2}{(1+f)^2}\leqslant 0. \qquad (3.17)
\end{aligned}
$$

We choose $0<\epsilon<\frac{1}{2}$ sufficiently small such that

$$
\tan(2\epsilon)\leqslant\frac{a_1}{160(1+a_1)}.
$$

We then have

$$
\frac{a_1}{10(1+a_1)}\left(g+2-4\frac{g'}{g}\right)^2\rho^2\frac{f_{,1}^4}{(1+f)^4}\geqslant\frac{a_1}{40(1+a_1)}g^2\rho^2\frac{f_{,1}^4}{(1+f)^4}\geqslant\tan(2\epsilon)4g^2\rho^2\frac{f_{,1}^4}{(1+f)^4}.
$$

By setting set $\eta=\frac{1}{\epsilon}$ we obtain

$$
\frac{a_1}{2(1+a_1)}\left(g+2-4\frac{g'}{g}\right)^2\rho^2\frac{f_{,1}^4}{(1+f)^4}
$$

$$
\begin{aligned}
&-\left(a_2+9\frac{a_1}{1+a_1}+n^2\frac{20(1+a_1)}{a_1}+2n^2+2n^2a_3\right)\Phi\rho\frac{f_{,1}^2}{(1+f)^2}\\
&-2n\rho\frac{f_{,1}^2}{(1+f)^2}-\left(n^2\frac{20(1+a_1)}{a_1}+2n^2a_3+\frac{4n^2}{(\ln(1+C)-\ln(1+f))^2}\right)\\
&-\frac{(n+2)^2}{8\delta}\frac{A}{\tan(\epsilon+\frac{\widetilde F}{A})}\exp\left\{\frac{m}{\ln(1+C)-\ln(1+f)}\right\}\rho\frac{f_{,1}^2}{(1+f)^2}+\frac{\widetilde F}{A}\tan\left(\epsilon+\frac{\widetilde F}{A}\right)b_0'(\alpha)\Phi\rho\frac{f_{,1}^2}{(1+f)^2}\leqslant 0.
\end{aligned}
\tag{3.18}
$$

We use the abbreviations:

$$
\begin{aligned}
&a:=\frac{a_1}{8(1+a_1)},\qquad b:=a_2+9\frac{a_1}{1+a_1}+n^2\frac{20(1+a_1)}{a_1}+2n^2+2n^2a_3,\\
&c:=n^2\frac{20(1+a_1)}{a_1}+2n^2a_3,
\end{aligned}
$$

and get the following form of the inequality:

$$
\begin{aligned}
&ag^2\rho^2\frac{f_{,1}^4}{(1+f)^4}+\left(\frac{\widetilde F}{A}\tan\left(\epsilon+\frac{\widetilde F}{A}\right)b_0'(\alpha)-b\right)\Phi\rho\frac{f_{,1}^2}{(1+f)^2}\\
&-\left(\frac{(n+2)^2}{8\delta}\frac{A}{\tan(\epsilon+\frac{\widetilde F}{A})}\exp\left\{\frac{m}{\ln(1+C)-\ln(1+f)}\right\}+2n\right)\rho\frac{f_{,1}^2}{(1+f)^2}\\
&-\left(c+\frac{4n^2}{(\ln(1+C)-\ln(1+f))^2}\right)\leqslant 0.
\end{aligned}
\tag{3.19}
$$

Multiply to both sides of (3.19) by $(\exp\{\frac{-m}{\ln(1+C)-\ln(1+f)}\})^2g^2$. Then we obtain

$$
\begin{aligned}
&a\left(\exp\left\{\frac{-m}{\ln(1+C)-\ln(1+f)}\right\}\right)^2g^4\rho^2\frac{f_{,1}^4}{(1+f)^4}\\
&+\left(\frac{\widetilde F}{A}\tan\left(\epsilon+\frac{\widetilde F}{A}\right)b_0'(\alpha)-b\right)\widetilde F\left(\exp\left\{\frac{-m}{\ln(1+C)-\ln(1+f)}\right\}\right)g^2\rho\frac{f_{,1}^2}{(1+f)^2}\\
&-\left(\frac{(n+2)^2}{8\delta}\frac{A}{\tan(\epsilon+\frac{\widetilde F}{A})}+2n\right)\exp\left\{\frac{-m}{\ln(1+C)-\ln(1+f)}\right\}g^2\rho\frac{f_{,1}^2}{(1+f)^2}\\
&-c16e^{-4}\frac{1}{m^2}-2916n^2e^{-6}\frac{1}{m^4}\leqslant 0.
\end{aligned}
\tag{3.20}
$$

Note that

$$
2\frac{(1-\delta)}{n-1}-3\frac{(n-2+\delta)^2}{n(n-1)(1-\delta)}=\frac{2}{n-1}\left(1-\delta-\frac{3}{2}\frac{(n-2+\delta)^2}{n(1-\delta)}\right).
$$

Since

$$
1-\delta-\frac{3}{2}\frac{(n-2+\delta)^2}{n(1-\delta)}=1-\delta-\frac{3}{4}\frac{\delta^2}{1-\delta}>0,
$$

for $n=2$ and $0<\delta<\frac{1}{3n-2}$ with

$$
\frac{\delta^2}{(1-\delta)^2}<\frac{4}{3},
$$

and

$$
1-\delta-\frac{3}{2}\frac{(n-2+\delta)^2}{n(1-\delta)}=1-\delta-\frac{(1+\delta)^2}{2(1-\delta)}>0,
$$

for $n=3$ and $0<\delta<\frac{1}{3n-2}$ with

$$\left(\frac{1+\delta}{1-\delta}\right)^2<2,$$

it follows that there exists a positive constant $K_1(n)>0$, such that $|\alpha|>K_1(n)$ implies that

$$\begin{aligned}b_0'(\alpha)=&\left(\frac{2(1-\delta)}{n-1}-3\frac{(n-2+\delta)^2}{n(n-1)(1-\delta)}\right)\alpha^2-2\frac{n-\delta}{n-1}\alpha+2-\frac{n^2+n\delta-2}{2(n-1)}\\&-\frac{(n-2)^2(n-1)}{8n}-\frac{3}{8}\frac{(n^2-n(1-2\delta)-2)^2}{n(n-1)(1-\delta)}>0.\end{aligned}$$

Denote $\widetilde{A}=\max_{M_C}\widetilde{F}$. Then by (3.20)

$$\begin{aligned}F\leqslant{}&\exp\{-\ln\cos(2\epsilon)\}\frac{1}{a}\left(\frac{(n+2)^2}{8\delta\epsilon\tan(\epsilon)}(\widetilde{A}+1)+2n+b\widetilde{A}\right)\\&+\exp\{-\ln\cos(2\epsilon)\}\left(\frac{1}{a}\left(16ce^{-4}\frac{1}{m^2}+2916n^2e^{-6}\frac{1}{m^4}\right)\right)^{\frac{1}{2}},\end{aligned}$$

which holds at p^*, where F attains its supremum. Hence, at any interior point of M_C, we have

$$\begin{aligned}&\exp\left\{\frac{-m}{\ln(1+C)-\ln(1+f)}\right\}g^2\rho\frac{f_{,1}^2}{(1+f)^2}\\&\quad\leqslant\exp\{-\ln\cos(2\epsilon)\}\frac{1}{a}\left(\frac{\eta(n+2)^2}{8\delta\tan(\epsilon)}(\widetilde{A}+1)+2n+b\widetilde{A}\right)\\&\qquad+\exp\{-\ln\cos(2\epsilon)\}\left(\frac{1}{a}\left(16ce^{-4}\frac{1}{m^2}+2916n^2e^{-6}\right)\right)^{\frac{1}{2}}.\end{aligned}\tag{3.21}$$

For $n\geqslant 4$, we have

$$1-\delta-\frac{3}{2}\frac{(n-2+\delta)^2}{n(1-\delta)}<0,$$

for $0<\delta<\frac{1}{3n-2}$ with

$$\frac{2n}{3}<\left(\frac{n-2+\delta}{1-\delta}\right)^2.$$

It follows that there exists a $K_2(n)>0$, such that $|\alpha|>K_2(n)$ implies that

$$\begin{aligned}b_0'(\alpha)=&\left(\frac{2(1-\delta)}{n-1}-3\frac{(n-2+\delta)^2}{n(n-1)(1-\delta)}\right)\alpha^2-2\frac{n-\delta}{n-1}\alpha+2-\frac{n^2+n\delta-2}{2(n-1)}\\&-\frac{(n-2)^2(n-1)}{8n}-\frac{3}{8}\frac{(n^2-n(1-2\delta)-2)^2}{n(n-1)(1-\delta)}<0.\end{aligned}$$

We choose ϵ sufficiently small such that

$$\tan(2\epsilon)\leqslant\min\left\{\frac{a_1}{160(1+a_1)},\frac{1}{(3+|\widetilde{a}|)}\frac{a\exp\{\ln\cos(1)\}\frac{1}{2(n-1)}}{3\frac{(n-2+\delta)^2}{n(n-1)(1-\delta)}-(1-\delta)\frac{2}{n-1}}\right\},$$

where

$$\widetilde{a}=\frac{n}{2(n-1)}-\frac{(n-2)^2}{n-1}-\frac{n^2-n-2}{8(n-1)}.$$

Then by (3.20)

$$
\begin{aligned}
& a\left(\exp\left\{\frac{-m}{\ln(1+C)-\ln(1+f)}\right\}\right)^2 g^4\rho^2\frac{f_{,1}^4}{(1+f)^4} \\
&\quad -\left(\frac{(n+2)^2}{8\delta}\frac{A}{\tan(\epsilon)}+2n+b\widetilde{A}\right)\exp\left\{\frac{-m}{\ln(1+C)-\ln(1+f)}\right\}g^2\rho\frac{f_{,1}^2}{(1+f)^2} \\
&\quad +\tan(2\epsilon)b_0'(\alpha)\widetilde{A}\exp\left\{\frac{-m}{\ln(1+C)-\ln(1+f)}\right\}g^2\rho\frac{f_{,1}^2}{(1+f)^2}-c16e^{-4}\frac{1}{m^2}-2916n^2e^{-6}\frac{1}{m^4}\leqslant 0. \qquad (3.22)
\end{aligned}
$$

Hence, at any interior point of M_C, we have

$$
\begin{aligned}
& \exp\left\{\frac{-m}{\ln(1+C)-\ln(1+f)}\right\}g^2\rho\frac{f_{,1}^2}{(1+f)^2} \\
&\quad \leqslant \exp\{-\ln\cos(2\epsilon)\}\frac{1}{a}\left(\frac{(n+2)^2}{8\delta\epsilon\tan(\epsilon)}(\widetilde{A}+1)+2n+b\widetilde{A}-\tan(2\epsilon)b_0'(\alpha)\widetilde{A}\right) \\
&\qquad +\exp\{-\ln\cos(2\epsilon)\}\left(\frac{1}{a}\left(16ce^{-4}\frac{1}{m^2}+2916n^2e^{-6}\frac{1}{m^4}\right)\right)^{\frac{1}{2}}. \qquad \square \qquad (3.23)
\end{aligned}
$$

4. Estimate for Φ

The purpose of this section is to show that Φ is bounded on M. To estimate Φ, we consider the function

$$
\widetilde{F}=\exp\left\{\frac{-m}{\ln(1+C)-\ln(1+f)}\right\}\Phi
$$

defined on M_C, where

$$
m=\frac{80(1+\frac{1}{n-1}-\delta\frac{3n-2}{2(n-1)})\ln(1+C)}{\frac{1}{n-1}-\delta\frac{3n-2}{2(n-1)}},\quad 0<\delta<\frac{1}{3n-2}.
$$

Obviously, $\widetilde{F}$ attains its supremum at some interior point p^*. We may assume that $\Phi(p^*)>0$. Then, at p^*,

$$
\frac{\Phi_{,i}}{\Phi}-g\frac{f_{,i}}{1+f}=0, \tag{4.1}
$$

$$
\frac{\Delta\Phi}{\Phi}-\sum\frac{\Phi_{,i}^2}{\Phi^2}-2g'\sum\frac{f_{,i}^2}{(1+f)^2}-g\frac{\Delta f}{1+f}+g\sum\frac{f_{,i}^2}{(1+f)^2}\leqslant 0. \tag{4.2}
$$

Inserting (4.1) into (4.2) we obtain

$$
\frac{\Delta\Phi}{\Phi}\leqslant 2g^2\frac{\|\nabla f\|_G^2}{(1+f)^2}+2g'\frac{\|\nabla f\|_G^2}{(1+f)^2}+g\frac{n}{(1+f)\rho}+\frac{n^2}{4}\frac{\Phi}{\rho}. \tag{4.3}
$$

On the other hand, by (2.7), we have

$$
\frac{\Delta\Phi}{\Phi}\geqslant\widetilde{a}g^2\sum\frac{f_{,i}^2}{(1+f)^2}+\widetilde{b}_0'(\alpha)\frac{\Phi}{\rho}, \tag{4.4}
$$

where we used (4.1) and

$$
\widetilde{a}=\frac{n}{2(n-1)}-\frac{(n-2)^2}{n-1}-\frac{n^2-n-2}{8(n-1)},
$$

$$
\widetilde{b}_0'(\alpha)=\frac{1}{n-1}\alpha^2-2\frac{n}{n-1}\alpha+2-\frac{(n-2)^2(n-1)}{8n}-\frac{n^2-2}{2(n-1)}-\frac{n^2-n-2}{2(n-1)}.
$$

Inserting (4.4) into (4.3) we get at p^*,

$$\widetilde{b}_0''(\alpha)\exp\left\{\frac{-m}{\ln(1+C)-\ln(1+f)}\right\}\Phi$$
$$\leqslant \big(3+|\widetilde{a}|\big)\exp\left\{\frac{-m}{\ln(1+C)-\ln(1+f)}\right\}g^2\rho\frac{\|\nabla f\|_G^2}{(1+f)^2}+4ne^{-2}\frac{1}{m}, \tag{4.5}$$

where

$$\widetilde{b}_0''(\alpha)=\widetilde{b}_0'(\alpha)-\frac{n^2}{4}.$$

We shall consider the following cases:

Case 1: $n=2,3$.

Case 2: $n\geqslant 4$.

Case 1: Note that

$$\widetilde{b}_0''(\alpha)=\frac{1}{n-1}\alpha^2-2\frac{n}{n-1}\alpha+2-\frac{(n-2)^2(n-1)}{8n}-\frac{n^2-2}{2(n-1)}-\frac{n^2-n-2}{2(n-1)}-\frac{n^2}{4}.$$

We can pick $K_3(n)>K_1(n)$ sufficiently large such that $|\alpha|>K_3(n)$ implies that

$$\widetilde{b}(\alpha)=\widetilde{b}_0''(\alpha)-\big(3+|\widetilde{a}|\big)\exp\big\{-\ln\cos(2\epsilon)\big\}\frac{1}{a}\left(\frac{(n+2)^2}{8\delta\epsilon\tan(\epsilon)}+b\right)>0.$$

Combining (3.15) and (4.5) we obtain the estimate

$$\widetilde{A}\leqslant\widetilde{Q}_1, \tag{4.6}$$

where

$$\widetilde{Q}_1=\frac{1}{\widetilde{b}(\alpha)}\big(3+|\widetilde{a}|\big)\exp\big\{-\ln\cos(2\epsilon)\big\}\frac{1}{a}\left(\frac{(n+2)^2}{8\delta\varepsilon\tan(\epsilon)}+2n\right)$$
$$+\frac{1}{\widetilde{b}(\alpha)}\left(\big(3+|\widetilde{a}|\big)\exp\big\{-\ln\cos(2\epsilon)\big\}\left(\frac{1}{a}\left(16ce^{-4}\frac{1}{m^2}+2916n^2e^{-6}\frac{1}{m^4}\right)\right)^{\frac{1}{2}}+4ne^{-2}\frac{1}{m}\right).$$

Hence, at any interior point of M_C we have

$$\Phi\leqslant\exp\left\{\frac{m}{\ln(1+C)-\ln(1+f)}\right\}\widetilde{Q}_1=\exp\left\{\frac{\frac{80(1+a_1)}{a_1}}{1-\frac{\ln(1+f)}{\ln(1+C)}}\right\}\widetilde{Q}_1.$$

Let $C\to\infty$,then

$$\Phi\leqslant\exp\left\{\frac{80(1+a_1)}{a_1}\right\}Q_1, \tag{4.7}$$

where

$$Q_1=\frac{1}{\widetilde{b}(\alpha)}\big(3+|\widetilde{a}|\big)\exp\big\{-\ln\cos(2\epsilon)\big\}\frac{1}{a}\left(\frac{(n+2)^2}{8\delta\varepsilon\tan(\epsilon)}+2n\right)$$

is a constant depending only on α and the dimension n.

Case 2: We set

$$\widetilde{b}(\alpha)=\left(\widetilde{b}_0''(\alpha)-\big(3+|\widetilde{a}|\big)\exp\big\{-\ln\cos(2\epsilon)\big\}\frac{1}{a}\left(\frac{(n+2)^2}{8\delta\epsilon\tan(\epsilon)}+b-\tan(2\epsilon)b_0'(\alpha)\right)\right).$$

Since

$$\big(3+|\widetilde{a}|\big)\exp\big\{-\ln\cos(1)\big\}\frac{1}{a}\tan(2\epsilon)\left(\frac{3(n-2+\delta)^2}{n(n-1)(1-\delta)}-\frac{2(1-\delta)}{n-1}\right)\leqslant\frac{1}{2(n-1)}.$$

We can pick $K_4(n) > K_2(n)$ sufficiently large such that $|\alpha| > K_4(n)$ implies that

$$\widetilde{b}(\alpha) = \widetilde{b}_0''(\alpha) + \bigl(3 + |\widetilde{a}|\bigr)\exp\bigl\{-\ln\cos(2\epsilon)\bigr\}\frac{1}{a}\tan(2\epsilon)b_0'(\alpha) - \bigl(3 + |\widetilde{a}|\bigr)\exp\bigl\{-\ln\cos(2\epsilon)\bigr\}\frac{1}{a}\left(\frac{(n+2)^2}{8\delta\epsilon\tan(\epsilon)} + b\right) > 0.$$

Then, by (3.17) and (4.5)

$$\widetilde{A} \leqslant \widetilde{Q}_2, \tag{4.8}$$

where

$$\widetilde{Q}_2 = \frac{1}{\widetilde{b}(\alpha)}\bigl(3 + |\widetilde{a}|\bigr)\exp\bigl\{-\ln\cos(2\epsilon)\bigr\}\frac{1}{a}\left(\frac{(n+2)^2}{8\delta\epsilon\tan(\epsilon)} + 2n\right) + \frac{1}{\widetilde{b}(\alpha)}\left(\Bigl(\bigl(3 + |\widetilde{a}|\bigr)\exp\bigl\{-\ln\cos(2\epsilon)\bigr\}\Bigl(\frac{1}{a}\Bigl(16ce^{-4}\frac{1}{m^2} + 2916n^2e^{-6}\frac{1}{m^4}\Bigr)\Bigr)\Bigr)^{\frac{1}{2}} + 4ne^{-2}\frac{1}{m}\right).$$

Therefore

$$\Phi \leqslant \exp\left\{\frac{80(1+a_1)}{a_1}\right\}Q_2, \tag{4.9}$$

where

$$Q_2 = \frac{1}{\widetilde{b}(\alpha)}\bigl(3 + |\widetilde{a}|\bigr)\exp\bigl\{-\ln\cos(2\epsilon)\bigr\}\frac{1}{a}\left(\frac{(n+2)^2}{8\delta\epsilon\tan(\epsilon)} + 2n\right)$$

is a constant depending only on α and the dimension n. □

5. Estimate for $\frac{1}{\rho}$

To estimate $\frac{1}{\rho}$ we consider the function

$$F^{(1)} = \exp\left\{\frac{-m}{\ln(1+C) - \ln(1+f)} + h\right\}\frac{1}{\rho}$$

defined on M_C, where $h = \epsilon_1\sum(\frac{\partial f}{\partial x_i})^2$, m and ϵ_1 are positive constants to be determined later. Clearly, $F^{(1)}$ attains its supremum at some interior point p^* of M_C. Choose a local orthonormal frame field $e_1, e_2, \ldots, e_n$ of the Blaschke metric G on M. Then we have, at p^*,

$$-g\frac{f_{,i}}{1+f} - \frac{\rho_{,i}}{\rho} + h_{,i} = 0, \tag{5.1}$$

$$-2g'\sum\frac{f_{,i}^2}{(1+f)^2} - g\frac{\Delta f}{1+f} + g\sum\frac{f_{,i}^2}{(1+f)^2} + \frac{\sum\rho_{,i}^2}{\rho^2} - \frac{\Delta\rho}{\rho} + \Delta h \leqslant 0. \tag{5.2}$$

Let us simplify (5.2). We note that

$$-g\frac{\Delta f}{1+f} = -g\frac{1}{1+f}\left(\frac{n}{\rho} + \frac{n}{\rho}\sum\rho_{,i}f_{,i}\right) = -g\frac{n}{(1+f)\rho} - g\frac{n}{(1+f)\rho}\sum\rho_{,i}f_{,i},$$

$$-\frac{\Delta\rho}{\rho} = -\alpha\sum\frac{\rho_{,i}^2}{\rho^2},$$

$$\Delta h = \frac{1}{\rho}\sum u^{ij}\frac{\partial^2 h}{\partial\xi_i\partial\xi_j} - \frac{2}{\rho^2}\sum u^{ij}\frac{\partial\rho}{\partial\xi_j}\frac{\partial h}{\partial\xi_i} = \frac{2\epsilon}{\rho}\sum u^{ii} - \frac{2}{\rho}\sum\rho_{,i}h_{,i}.$$

Hence, by (5.2)

$$-2g'\sum\frac{f_{,i}^2}{(1+f)^2}-g\frac{n}{(1+f)\rho}-g\frac{n}{(1+f)\rho}\sum\rho_{,i}f_{,i}$$
$$+g\sum\frac{f_{,i}^2}{(1+f)^2}+\frac{\sum\rho_{,i}^2}{\rho^2}-\alpha\sum\frac{\rho_{,i}^2}{\rho^2}+\frac{2\epsilon}{\rho}\sum u^{ii}-\frac{2}{\rho}\sum\rho_{,i}h_{,i}\leqslant 0. \tag{5.3}$$

Inserting (5.1) into (5.3) and applying the Cauchy–Schwarz inequality we obtain

$$-2g'\sum\frac{f_{,i}^2}{(1+f)^2}-g\frac{n}{(1+f)\rho}+\left(n-\frac{1}{2}\right)g^2\sum\frac{f_{,i}^2}{(1+f)^2}-\frac{n^2+2}{2}\sum h_{,i}^2$$
$$+g\sum\frac{f_{,i}^2}{(1+f)^2}-\alpha\sum\frac{\rho_{,i}^2}{\rho^2}+\frac{2\epsilon}{\rho}\sum u^{ii}\leqslant 0. \tag{5.4}$$

We choose the following values for ϵ and m:

$$m=\frac{80(1+\frac{1}{n-1}-\delta\frac{3n-2}{2(n-1)})\ln(1+C)}{\frac{1}{n-1}-\delta\frac{3n-2}{2(n-1)}},\quad 0<\delta<\frac{1}{3n-2},$$
$$\epsilon_1=\frac{1}{2(n^2+2)\max_{M_C}\sum(\frac{\partial f}{\partial x_i})^2}.$$

Then we have

$$\frac{1}{2}g^2\sum\frac{f_{,i}^2}{(1+f)^2}\geqslant 2g'\sum\frac{f_{,i}^2}{(1+f)^2}, \tag{5.5}$$
$$\frac{n^2+2}{2}\sum h_{,i}^2=\frac{n^2+2}{2}\epsilon_1^2\sum\left(\left(\sum\xi_j^2\right)_{,i}\right)^2$$
$$\leqslant\epsilon_1^2\frac{n^2+2}{2}\frac{4}{\rho}\sum u^{ii}\sum\xi_i^2=\epsilon_1^2\frac{2(n^2+2)}{\rho}\sum u^{ii}\sum\xi_i^2. \tag{5.6}$$

Inserting (5.5) and (5.6) into (5.4), we get

$$\sum u^{ii}\leqslant\frac{n}{\epsilon_1}g+\frac{1}{\epsilon_1}|\alpha|\sum\frac{\rho_{,i}^2}{\rho}. \tag{5.7}$$

Denote by $\lambda_1,\dots,\lambda_n$ the eigenvalues of the matrix (f_{ij}). Recall that $\frac{1}{\rho}=(\det(f_{ij}))^{\frac{1}{n+2}}$. Then

$$\frac{1}{\rho^{\frac{n+2}{n}}}\leqslant\lambda_1+\cdots+\lambda_n\leqslant\sum u^{ii}.$$

Consequently, it follows from (5.7) that

$$\exp\left\{\frac{-m}{\ln(1+C)-\ln(1+f)}+h\right\}\frac{1}{\rho}\leqslant\exp\left\{\frac{1}{2(n^2+2)}\right\}\left(\frac{4n}{\epsilon_1}\exp\{-2\}\frac{1}{m}+\frac{1}{\epsilon_1}|\alpha|Q\right)^{\frac{n}{n+2}}, \tag{5.8}$$

where $Q=\max\{\widetilde{Q}_1,\widetilde{Q}_2\}$. Inequality (5.8) holds at p^*, where $F^{(1)}$ attains its supremum. Thus we get

Proposition 5.1. *At any interior point of M_C, we have*

$$\exp\left\{\frac{-m}{\ln(1+C)-\ln(1+f)}\right\}\frac{1}{\rho}\leqslant\exp\left\{\frac{1}{2(n^2+2)}\right\}\left(\frac{4n}{\epsilon_1}\exp\{-2\}\frac{1}{m}+\frac{1}{\epsilon_1}|\alpha|Q\right)^{\frac{n}{n+2}} \tag{5.9}$$

where Q is a constant depending only on the dimension n.

6. Estimate for ρ

To estimate ρ, we consider the function

$$F^{(2)} = \exp\left\{\frac{-m}{\ln(1+C) - \ln(1+f)} + h\right\}\rho$$

defined on M_C, where $h = \epsilon_2 \sum x_i^2$, m and ϵ_2 are positive constants to be determined later. Clearly, $F^{(2)}$ attains its supremum at some interior point p^* of M_C. Choose a local orthonormal frame field $e_1, e_2, \ldots, e_n$ of the Blaschke metric G on M. Then we have, at p^*,

$$-g\frac{f_{,i}}{1+f} + \frac{\rho_{,i}}{\rho} + h_{,i} = 0, \tag{6.1}$$

$$-2g'\sum \frac{f_{,i}^2}{(1+f)^2} - g\frac{\Delta f}{1+f} + g\sum\frac{f_{,i}^2}{(1+f)^2} - \sum\frac{\rho_{,i}^2}{\rho^2} + \frac{\Delta\rho}{\rho} + \Delta h \leqslant 0. \tag{6.2}$$

Inserting (6.1) into (6.2) we obtain

$$\begin{aligned}&-2g'\sum \frac{f_{,i}^2}{(1+f)^2} - g\frac{n}{(1+f)\rho} - \left(n+\frac{1}{2}\right)g^2\sum\frac{f_{,i}^2}{(1+f)^2} - \frac{3}{4}n^2\sum h_{,i}^2\\ &\quad + g\sum\frac{f_{,i}^2}{(1+f)^2} + (\alpha-2)\sum\frac{\rho_{,i}^2}{\rho^2} + \frac{2\epsilon}{\rho}\sum f^{ii} \leqslant 0.\end{aligned} \tag{6.3}$$

We choose the following values for m and ϵ_2:

$$m = \frac{80\big(1+\frac{1}{n-1} - \delta\frac{3n-2}{2(n-1)}\big)\ln(1+C)}{\frac{1}{n-1} - \delta\frac{3n-2}{2(n-1)}}, \quad 0<\delta<\frac{1}{3n-2},$$

$$\epsilon_2 = \frac{1}{(3n^2+8(n+1))\max_{M_C}\sum x_i^2}.$$

Then we have

$$\frac{1}{2}g^2\sum\frac{f_{,i}^2}{(1+f)^2} \geqslant 2g'\sum\frac{f_{,i}^2}{(1+f)^2}. \tag{6.4}$$

Note that

$$g^2\sum\frac{f_{,i}^2}{(1+f)^2} = \sum\left(\frac{\rho_{,i}}{\rho} + h_{,i}\right)^2 \leqslant 2\sum\frac{\rho_{,i}^2}{\rho^2} + 2\sum h_{,i}^2, \tag{6.5}$$

$$\sum h_{,i}^2 \leqslant \epsilon_2^2\frac{4}{\rho}\sum f^{ii}\sum x_i^2. \tag{6.6}$$

Inserting (6.4), (6.5) and (6.6) into (6.3) we obtain the following estimate:

$$\sum f^{ii} \leqslant \frac{n}{\epsilon_2}g + \frac{1}{\epsilon_2}\big(2(n+2)+|\alpha|\big)\Phi. \tag{6.7}$$

Denote by $\lambda_1, \ldots, \lambda_n$ the eigenvalues of the matrix (u_{ij}). Recall that $\rho = (\det(u_{ij}))^{\frac{1}{n+2}}$. Then

$$\rho^{\frac{n+2}{n}} = \sqrt[n]{\lambda_1\cdots\lambda_n} \leqslant \frac{\lambda_1+\cdots+\lambda_n}{n} \leqslant \lambda_1+\cdots+\lambda_n = \sum f^{ii}.$$

Consequently, it follows from (6.7) that

$$\begin{aligned}&\exp\left\{\frac{-m}{\ln(1+C)-\ln(1+f)} + h\right\}\rho\\ &\quad\leqslant \exp\left\{\frac{1}{3n^2+8(n+1)}\right\}\left(\frac{4n}{\epsilon_2}\exp\{-2\}\frac{1}{m} + \frac{1}{\epsilon_2}\big(2(n+2)+|\alpha|\big)Q\right)^{\frac{n}{n+2}}.\end{aligned} \tag{6.8}$$

Inequality (6.8) holds at p^*, where $F^{(2)}$ attains its supremum. Hence, at any interior point of M_C, we have

$$\begin{aligned}&\exp\left\{\frac{-m}{\ln(1+C)-\ln(1+f)}\right\}\rho\\&\quad\leqslant \exp\left\{\frac{1}{3n^2+8(n+1)}\right\}\left(\frac{4n}{\epsilon_2}\exp\{-2\}\frac{1}{m}+\frac{1}{\epsilon_2}\big(2(n+2)+|\alpha|\big)Q\right)^{\frac{n}{n+2}}. \qquad \square \end{aligned} \tag{6.9}$$

7. Proof of Theorem 1

Let $\Omega\subset\mathbb{R}^n$ be a bounded convex domain and $f\in C^1(\Omega)\cap C^0(\overline{\Omega})$ an nonnegative convex function in Ω such that $f\leqslant T$ on $\partial\Omega$, where T is a positive constant. If $x\in\Omega$, then (see [5])

$$\sqrt{\sum\left(\frac{\partial f}{\partial x_i}\right)^2(x)}\leqslant\frac{T}{\operatorname{dist}(x,\partial\Omega)}. \tag{7.1}$$

Let $f\in C^4(\Omega)\cap C^0(\overline{\Omega})$ be an nonnegative strict convex solution to the problem

$$\begin{aligned}&\sum U^{ij}\frac{\partial^2}{\partial x_i\partial x_j}\left(\left(\det\left(\frac{\partial^2 f}{\partial x_i\partial x_j}\right)\right)^{-\frac{n+1-\alpha}{n+2}}\right)=0,\quad \alpha\neq n+1,\ \text{in } \Omega,\\&f=T,\quad \text{on } \partial\Omega.\end{aligned}$$

Let $\dot{x}\in\Omega$ such that $f(\dot{x})=\min_{\overline{\Omega}} f$. By the unimodular affine transformation:

$$\begin{aligned}&\widetilde{x}_i=x_i-\dot{x}_i\quad(1\leqslant i\leqslant n),\\&\widetilde{x}_{n+1}=-\sum\frac{\partial f}{\partial x_i}(\dot{x}_1,\dots,\dot{x}_n)(x_i-\dot{x}_i)+x_{n+1}-f(\dot{x}_1,\dots,\dot{x}_n),\end{aligned}$$

we may assume that $\dot{x}$ has coordinates $(0,\dots,0)$ and

$$f(0)=0,\quad \frac{\partial f}{\partial x_i}(0)=0,\quad \text{for all } i\geqslant 1.$$

Given $0<T'<T$ and let $\Omega_{T'}=\{x\in\Omega\mid f(x)<T'\}$. By (5.9), (6.9) and (7.1), there exists a constant $K=K(n)$ depending only on the dimension n such that, if $|\alpha|\geqslant K(n)$, then we have the estimate

$$0<C_1\leqslant\det\left(\frac{\partial^2 f}{\partial x_i\partial x_j}\right)\leqslant C_2,\quad \text{in } \Omega_{T'+\frac{T-T'}{4}}, \tag{7.2}$$

where C_1 and C_2 are constants depending only on T, T', $\operatorname{diam}(\overline{\Omega_{T'+\frac{T-T'}{2}}})$, $\operatorname{dist}(\overline{\Omega_{T'+\frac{T-T'}{2}}},\partial\Omega)$, α and the dimension n. By (7.2) we may use the Caffarelli–Gutierrez theory (see [6]) to obtain a Hölder estimate for the function

$$w=\left(\det\left(\frac{\partial^2 f}{\partial x_i\partial x_j}\right)\right)^{-\frac{n+1-\alpha}{n+2}},$$

namely,

$$[w]_{r,\Omega_{T'}}\leqslant N,$$

for any $0<T'<T$, where r and N are positive constants depending only on T, T', $\operatorname{diam}(\overline{\Omega_{T'+\frac{T-T'}{2}}})$, $\operatorname{dist}(\overline{\Omega_{T'+\frac{T-T'}{2}}},\partial\Omega)$, α and the dimension n. Then we use the Caffarelli–Schauder estimate for the Monge–Ampère equation (see [7]) to get a local $C^{2,r}$ estimate for the function f,

$$|f|_{2,r,\Omega_{T'}}\leqslant N,$$

where again r and N are positive constants depending only on T, T', $\operatorname{diam}(\overline{\Omega_{T'+\frac{T-T'}{2}}})$, $\operatorname{dist}(\overline{\Omega_{T'+\frac{T-T'}{2}}},\partial\Omega)$, α and the dimension n. Finally, by bootstrapping, the Theorem 1 follows. $\square$

8. Proof of Theorem 2

The key point of the proof of Theorem 2 is to estimate Φ. We shall show that $\Phi \equiv 0$ on M everywhere. Therefore $\det(\frac{\partial^2 f}{\partial x_i \partial x_j}) = \text{const}$ on M. Then Theorem 2 follows by E. Calabi's Theorem. Choose a sequence $\{\lambda_k\}$ of positive numbers such that $\lambda_k \to \infty$ as $k \to \infty$. We let

$$f^{(k)} = \frac{1}{\lambda_k} f, \rho^{(k)} = \left(\det\left(\frac{\partial^2 f^{(k)}}{\partial x_i \partial x_j}\right)\right)^{-1/(n+2)},$$
$$\Phi^{(k)} = \frac{\|\nabla \rho^{(k)}\|^2_{G^{(k)}}}{\rho^{(k)}}, \quad k = 1, 2, 3, \ldots, \tag{8.1}$$

where $G^{(k)}$ denotes the Blaschke metric of the locally strongly convex hypersurface

$$M^{(k)} = \left\{\left(x_1, \ldots, x_n, f^{(k)}(x_1, x_2, \ldots, x_n)\right) \mid (x_1 x_2, \ldots, x_n) \in R^n\right\}.$$

We have

$$\Phi^{(k)}(x_1, x_2, \ldots, x_n) = \frac{1}{\rho^{(k)}} \sum G^{(k)ij} \frac{\partial \rho^{(k)}}{\partial x_i} \frac{\partial \rho^{(k)}}{\partial x_j}(x_1, x_2, \ldots, x_n) = \lambda_k \Phi(x_1, x_2, \ldots, x_n). \tag{8.2}$$

Since $f^{(k)}$ satisfies the PDE (0.1), from (4.7) and (4.9) it follows that

$$\Phi^{(k)} \leqslant Q, \quad k = 1, 2, 3, \ldots, \tag{8.3}$$

where Q is a constant depending only on α and the dimension n. On the other hand, for any $(\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n) \in \mathbb{R}^n$, if $\Phi(\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n) \neq 0$, we have

$$\Phi^{(k)}(\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n) = \lambda_k \Phi(\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n) \to \infty, \quad \text{as } k \to \infty.$$

This contradicts (8.3). Thus

$$\Phi \equiv 0.$$

Consequently

$$\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) = \text{const} > 0.$$

This means that M is an Euclidean complete affine hypersphere. By a theorem of S.Y. Cheng and S.T. Yau (see [9]), M is an affine complete parabolic affine hypersphere. Then, by a result of E. Calabi (see [4, p. 128]) we conclude that M must be elliptic paraboloid. This complete the proof of the Theorem 2. □

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